Generalization of He's Exp-Function Method and New Exact Solutions for Burgers Equation

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A generalization of He's Exp-function method for nonlinear equations is introduced in this research. New exact solutions are obtained for Burgers equation.

Key words: Exp-Function Method; Soliton; Burgers Equation; Nonlinear Equations.

1. Introduction

Recently, He and Wu [1] proposed a straightforward and concise method, called Exp-function method for solving nonlinear equations. The method has attracted much attention and has been used by many authors [2 – 13] to obtain travelling and nontravelling wave solutions, compact-like solutions, and periodic solutions of various nonlinear wave equations. In [5], we showed that the main advantage of this method over the other methods [14 – 23] is that it can be applied to a wide class of nonlinear evolution equations including those in which the odd and even-order derivative terms are coexist. Also in [6], we used the method to obtain different types of exact solutions for the generalized Klein-Gordon equation.

In [7] and [8], Sheng Zhang applied the method to obtain exact solutions for the Korteweg-de Vries (KdV) equation with variable coefficients and Maccari's system, respectively. Also in [9], he applied the method to obtain generalized solitonary solutions for Riccati equation and hence new exact solutions with three arbitrary functions were obtained for the $(2+1)$-dimensional Broer-Kaup-Kupershmidt equations. In [24, 25], Zhu extended the method to solve differential-difference equations. In the review article [26], He introduced a new interpretation of the method. Also in [27], Zhou et. al., suggested a modified version of the Exp-function method for nonlinear equation with high order of nonlinearity.

Although most of the research effort has been focused on using the method to solve different kinds of nonlinear equations as mentioned above, no further improvement has been done on the method till this moment. In this paper, the Exp-function method is generalized and new exact soliton solutions for Burgers equation are obtained. We also showed in this work that the main advantage of the generalized Exp-function method is that it can be applied to nonlinear equations including those in which the odd and even-order derivative terms are coexist. Burgers equation is chosen to illustrate the method of solution.

2. Basic Idea of the Generalized Exp-Function Method

Consider a given nonlinear wave equation

$$N(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \dots) = 0, \tag{1}$$

we seek its wave solutions

$$u = u(\eta), \quad \eta = k(x - \lambda t). \tag{2}$$

Consequently, (1) is reduced to the ordinary differential equation (ODE):

$$N(u, -k\lambda u', ku', k^2u'', k^2\lambda^2 u'', -k^2\lambda u'', \dots) = 0. \tag{3}$$

The generalized Exp-function method is based on the assumption that the travelling wave solutions can be expressed in the following form:

$$\begin{aligned} u(\eta) &= \frac{\sum_{n=-c}^{p} a_n\, [\phi(\eta)]^n}{\sum_{m=-d}^{q} b_m\, [\phi(\eta)]^m} \\ &= \frac{a_{-c}[\phi(\eta)]^{-c} + \cdots + a_p[\phi(\eta)]^p}{b_{-d}[\phi(\eta)]^{-d} + \cdots + b_q[\phi(\eta)]^q}, \end{aligned} \tag{4}$$

Case	A	B	C	$\phi(\eta)$: Solution of Riccati Equation (5)
1	free	$\neq 0$	0	$-\frac{A}{B}+\frac{1}{B}\mathrm{e}^{(B\eta)}$, $\mathrm{i}^2=-1$
2	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{\tanh(\eta)}{1\pm\mathrm{sech}(\eta)}$, $\coth(\eta)\pm\mathrm{csch}(\eta)$, $\frac{\coth(\eta)}{1\pm\mathrm{icsch}(\eta)}$, $\tanh(\eta)\pm\mathrm{isech}(\eta)$
3	$\frac{1}{2}$	0	$\frac{1}{2}$	$\csc(\eta)-\cot(\eta)$, $\tan(\eta)\pm\sec(\eta)$, $\frac{\tan(\eta)}{1\pm\sec(\eta)}$
4	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\sec(\eta)-\tan(\eta)$, $\cot(\eta)\pm\csc(\eta)$, $\frac{\cot(\eta)}{1\pm\csc(\eta)}$
5	1	0	-1	$\tanh(\eta)$, $\coth(\eta)$
6	1	0	1	$\tan(\eta)$
7	-1	0	-1	$\cot(\eta)$
8	1	-2	2	$\frac{\tan(\eta)}{1+\tan(\eta)}$
9	1	2	2	$\frac{\tan(\eta)}{1-\tan(\eta)}$
10	-1	2	-2	$\frac{\cot(\eta)}{1+\cot(\eta)}$
11	-1	-2	-2	$\frac{\cot(\eta)}{1-\cot(\eta)}$
12	0	0	$\neq 0$	$\frac{-1}{C\eta+C_0}$
13	$\neq 0$	0	0	$A\eta+C_0$

Table 1. The exact solutions for Riccati Equation (5).

where c, d, p and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. In addition, $\phi(\eta)$ satisfies Riccati equation

$$\phi'(\eta)=A+B\phi(\eta)+C\phi^2(\eta). \tag{5}$$

3. Analysis of the Method

In this paper, we aim to take the full advantage of Riccati equation and use its solutions together with the ansatz (4) to construct new exact solutions for nonlinear equations. Riccati equation (5) has many exact solutions introduced in [28] by Chen and Zhang except case (13) and given in the following Table 1.

4. Application of the Generalized Exp-Function Method to Burgers Equation

A well-known model is the one-dimensional Burgers equation

$$u_t+uu_x-\nu u_{xx}=0, \tag{6}$$

where $\nu>0$ is the coefficient of the kinematics viscosity of the fluid. This equation was formulated by Burgers in an attempt to model turbulent flow in a channel. Using the transformation (2), (6) becomes

$$-\lambda u'+uu'-\nu ku''=0. \tag{7}$$

On integrating this equation once with reference to η and assuming that the constant of integration is zero, we get

$$-\lambda u+\frac{u^2}{2}-\nu ku'=0. \tag{8}$$

Using ansatz (4) and Riccati equation (5) for the highest linear term u' with the highest order nonlinear term u^2, we have

$$u'=\frac{a_1\phi^{(-c-d-1)}+\cdots+a_2\phi^{(p+q+1)}}{b_1\phi^{(-2d)}+\cdots+b_2\phi^{(2q)}} \tag{9}$$

and

$$u^2=\frac{a_3\phi^{(-2c)}+\cdots+a_4\phi^{(2p)}}{b_3\phi^{(-2d)}+\cdots+b_4\phi^{(2q)}}, \tag{10}$$

where a_i and b_i are determined coefficients only for simplicity. From balancing the lowest order and highest order of ϕ in (9) and (10), we obtain $-c-d-1=-2c$ and $p+q+1=2p$ which give $c=d+1$ and $p=q+1$. Here, we only consider the simplest case $d=q=1$ and consequently $c=p=2$. Now the ansatz (4) becomes

$$\begin{aligned}u(\eta)&=\frac{a_{-2}\phi^{-2}+a_{-1}\phi^{-1}+a_0+a_1\phi+a_2\phi^2}{b_{-1}\phi^{-1}+b_0+b_1\phi}\\&=\frac{a_{-2}+a_{-1}\phi+a_0\phi^2+a_1\phi^3+a_2\phi^4}{b_{-1}\phi+b_0\phi^2+b_1\phi^3}.\end{aligned} \tag{11}$$

Substituting (11) into (8) and multiplying the resulting equation by $(b_{-1}\phi+b_0\phi^2+b_1\phi^3)^2$, we obtain

$$\begin{aligned}&\delta_0+\delta_1\phi+\delta_2\phi^2+\delta_3\phi^3+\delta_4\phi^4\\&+\delta_5\phi^5+\delta_6\phi^6+\delta_7\phi^7+\delta_8\phi^8=0,\end{aligned} \tag{12}$$

where

$$\delta_0=\frac{1}{2}(a_{-2}^2+2\nu kA),$$

$$\delta_1=2[a_{-1}-(\lambda-\nu Bk)b_{-1}+2\nu kAb_0]a_{-2},$$

$$\begin{aligned}\delta_2 = a_{-1}^2 + 2a_0a_{-2} + 2\nu kCa_{-2}b_{-1} - 2\lambda a_{-1}b_{-1}\\ - 2\nu kAa_0b_{-1} - 2(\lambda - 2\nu Bk)a_{-2}b_0\\ + 2\nu kAa_{-1}b_0 + 6\nu kAa_{-2}b_1,\end{aligned}$$

$$\begin{aligned}\delta_3 = 2a_0a_{-1} + 2a_1a_{-2} - 2(\lambda + 2\nu Bk)a_0b_{-1}\\ - 4\nu kAa_1b_{-1} + 4\nu kCa_{-2}b_0\\ - 2(\lambda - \nu kB)a_{-1}b_0 - 2(\lambda - 3\nu kB)a_{-2}b_1\\ + 4\nu kAa_{-1}b_1 + 8\nu kAa_{-2}b_2,\end{aligned}$$

$$\begin{aligned}\delta_4 = a_0^2 + 2a_1a_{-1} - 2\nu kCa_0b_{-1}\\ - 2(\lambda + 2\nu Bk)a_1b_{-1} + 2\nu kCa_{-1}b_0\\ - 2\lambda a_0b_0 - 2\nu kAa_1b_0 + 6\nu kCa_{-2}b_1\\ - 2(\lambda - 2\nu Bk)a_{-1}b_1 + 2\nu kAa_0b_1\\ - 2(\lambda - 4\nu Bk)a_{-2}b_2 + 6\nu kAa_{-1}b_2,\end{aligned}$$

$$\begin{aligned}\delta_5 = 2a_0a_1 - 4\nu kCa_1b_{-1}\\ - 2(\lambda + \nu kB)a_1b_0 - 2(\lambda - \nu kB)a_0b_1\\ + 4\nu kCa_{-1}b_1 + 8\nu kCa_{-2}b_2\\ - 2(\lambda - 3\nu kB)a_{-1}b_2 + 4\nu kAa_0b_2,\end{aligned}$$

$$\begin{aligned}\delta_6 = a_1^2 - 2\nu kCa_1b_0 + 2\nu kCa_0b_1\\ - 2\lambda a_1b_1 + 6\nu kCa_{-1}b_2\\ - 2(\lambda - 2\nu kB)a_0b_2 + 2\nu kCa_1b_2,\end{aligned}$$

$$\delta_7 = 4\nu kCa_0b_2 - 2(\lambda - \nu kB)a_1b_2,$$

$$\delta_8 = -2\nu kCa_1b_2.$$

By setting

$$\delta_0 = \delta_1 = \cdots = \delta_8 = 0 \tag{13}$$

and solving this system of algebraic equations by using MATHEMATICA, we obtain the following results:

Case A.

$$\begin{aligned}&a_{-2} = 0,\ a_2 = 0,\ a_1 = 0,\ b_1 = 0,\ b_{-1} = 0,\\ &\lambda = \pm\nu k\sqrt{B^2 - 4AC},\ b_0 = -\frac{a_{-1}}{2\nu kA},\\ &a_0 = \frac{1}{2A}\left(Ba_{-1} \pm |a_{-1}|\sqrt{B^2 - 4AC}\right).\end{aligned} \tag{14}$$

Case B.

$$\begin{aligned}&a_{-2} = 0,\ a_2 = 0,\ a_{-1} = 0,\ b_1 = 0,\ b_{-1} = 0,\\ &\lambda = \frac{\pm k\nu|a_1|\sqrt{B^2 - 4AC}}{a_1},\ b_0 = \frac{a_1}{2\nu kC},\\ &a_0 = \frac{Ba_1 \pm |a_1|\sqrt{B^2 - 4AC}}{a_1}.\end{aligned} \tag{15}$$

Case C.

$$\begin{aligned}&a_{-2} = 0,\ a_2 = 0,\ a_1 = 0,\ b_1 = 0,\\ &a_0 = \frac{1}{2A}(Ba_{-1} + \zeta_1),\ \zeta_1 = |a_{-1}|\sqrt{B^2 - 4AC},\\ &b_0 = \frac{1}{4\nu kA^2}\Big(-Aa_{-1} + 2\nu kABb_{-1}\\ &\quad \pm |A|\sqrt{a_{-1}^2 + 4\nu kb_{-1}\left[\nu k(B^2 - 4AC)b_{-1} + \zeta_1\right]}\Big),\\ &\lambda = \frac{1}{4Aa_{-1}b_{-1}}\Big(Aa_{-1}^2 - 2\nu kAb_{-1}\zeta_1 \pm a_{-1}|A|\\ &\quad \cdot \sqrt{a_{-1}^2 + 4\nu kb_{-1}\left[\nu k(B^2 - 4AC)b_{-1} + \zeta_1\right]}\Big).\end{aligned} \tag{16}$$

Case D.

$$\begin{aligned}&a_{-2} = 0,\quad a_2 = 0,\quad a_1 = 0,\quad b_1 = 0,\\ &a_0 = \frac{1}{2A}(Ba_{-1} - \zeta_1),\\ &b_0 = \frac{1}{4\nu kA^2}\Big(-Aa_{-1} + 2\nu kABb_{-1}\\ &\quad \pm |A|\sqrt{a_{-1}^2 + 4\nu kb_{-1}[\nu k(B^2 - 4AC)b_{-1} - \zeta_1]}\Big),\\ &\lambda = \frac{1}{4Aa_{-1}b_{-1}}\Big(Aa_{-1}^2 + 2\nu kAb_{-1}\zeta_1 \pm a_{-1}|A|\\ &\quad \cdot \sqrt{a_{-1}^2 + 4\nu kb_{-1}[\nu k(B^2 - 4AC)b_{-1} - \zeta_1]}\Big).\end{aligned} \tag{17}$$

Case E.

$$\begin{aligned}&a_{-2} = 0,\quad a_2 = 0,\quad a_0 = 0,\\ &b_{-1} = \frac{\pm Ba_{-1}\,(2k\nu Ab_0 + a_{-1})}{2k\nu|B|\zeta_1},\\ &a_1 = \frac{-(B^2 - 2AC)a_{-1} \pm \zeta_1|B|}{2A^2},\\ &b_1 = \big[B^2(B^2 - 4AC)a_{-1}^2 \mp ((B^2 - 2AC)a_{-1}\\ &\quad + 4\nu kA^2Cb_0)\zeta_1|B|\big]\left[4\nu kA^2B(B^2 - 4AC)a_{-1}\right]^{-1},\\ &\lambda = \frac{\pm k\nu|B|\zeta_1}{Ba_{-1}}.\end{aligned} \tag{18}$$

Case F.

$$\begin{aligned}&a_{-2} = 0,\quad a_2 = 0,\quad a_{-1} = 0,\quad b_{-1} = 0,\\ &a_0 = \frac{1}{2C}(Ba_1 - \zeta_2),\quad \zeta_2 = |a_1|\sqrt{B^2 - 4AC},\\ &b_0 = \frac{1}{4\nu kC^2}\Big(Ca_1 + 2\nu kABb_1\\ &\quad \pm |C|\sqrt{a_1^2 + 4\nu kb_1[\nu k(B^2 - 4AC)b_1 + \zeta_2]}\Big),\end{aligned}$$

$$\lambda = \frac{1}{4a_1b_1C}\left(Ca_1^2 - 2\nu kCb_1\zeta_2 \mp a_1|C|\sqrt{a_1^2+4\nu kb_1[\nu k(B^2-4AC)b_1+\zeta_2]}\right). \quad (19)$$

Substituting the results of case A into (11) and using the special solutions of (5), we obtain the following multiple soliton-like and triangular periodic solutions for the nonlinear Burgers equation:

$$u_1 = 2\nu kB\left(-1+\frac{A}{A-\mathrm{e}^{B\eta}}\right), \quad \lambda = -\nu kB, \quad (20)$$

$$u_2 = -\nu k\left(1+\frac{1\pm\operatorname{sech}(\eta)}{\tanh(\eta)}\right), \quad \lambda = -\nu k, \quad (21)$$

$$u_3 = -\nu k\left(1+\frac{1\pm\mathrm{i}\operatorname{csch}(\eta)}{\coth(\eta)}\right), \quad \lambda = -\nu k, \quad (22)$$

$$u_4 = -\nu k\left(1+\frac{1}{\coth(\eta)\pm\operatorname{csch}(\eta)}\right), \quad \lambda = -\nu k, \quad (23)$$

$$u_5 = -\nu k\left(1+\frac{1}{\tanh(\eta)\pm\mathrm{i}\operatorname{sech}(\eta)}\right), \quad \lambda = -\nu k, \quad (24)$$

$$u_6 = -\nu k\left(\mathrm{i}+\frac{1}{\csc(\eta)-\cot(\eta)}\right), \quad \lambda = -\mathrm{i}\nu k, \quad (25)$$

$$u_7 = -\nu k\left(\mathrm{i}+\frac{1}{\tan(\eta)\pm\sec(\eta)}\right), \quad \lambda = -\mathrm{i}\nu k, \quad (26)$$

$$u_8 = -\nu k\left(\mathrm{i}+\frac{1\pm\sec(\eta)}{\tan(\eta)}\right), \quad \lambda = -\mathrm{i}\nu k, \quad (27)$$

$$u_9 = \nu k\left(\mathrm{i}+\frac{1}{\sec(\eta)-\tan(\eta)}\right), \quad \lambda = \mathrm{i}\nu k, \quad (28)$$

$$u_{10} = \nu k\left(\mathrm{i}+\frac{1}{\cot(\eta)\pm\csc(\eta)}\right), \quad \lambda = \mathrm{i}\nu k, \quad (29)$$

$$u_{11} = \nu k\left(\mathrm{i}+\frac{1\pm\csc(\eta)}{\cot(\eta)}\right), \quad \lambda = \mathrm{i}\nu k, \quad (30)$$

$$u_{12} = -2\nu k[1+\tanh(\eta)], \quad \lambda = -2\nu k, \quad (31)$$

$$u_{13} = -2\nu k[1+\coth(\eta)], \quad \lambda = -2\nu k, \quad (32)$$

$$u_{14} = -2\nu k[\mathrm{i}\pm\cot(\eta)], \quad \lambda = -2\mathrm{i}\nu k, \quad (33)$$

$$u_{15} = 2\nu k[-\mathrm{i}+\tan(\eta)], \quad \lambda = -2\mathrm{i}\nu k, \quad (34)$$

$$u_{16} = \frac{-2\nu kA}{kAx+C_0}. \quad (35)$$

To compare our results with those obtained in [5], we find that the exact solution given by u_{12} can be written as

$$u_{12} = -2\nu k[1+\tanh(k(x-\lambda t))] = \lambda\left[1-\tanh(\frac{\lambda}{2\nu}(x-\lambda t))\right], \quad (36)$$

with $\lambda = -2\nu k$, which is the travelling wave solution obtained by the Exp-function method in [5], Eq. 31. Moreover, all the exact solutions obtained above can be verified by substitution. It should be noted that the exact solutions obtained above were derived from the results of case A only with the special solutions of Riccati equation. So, we may obtain more exact solutions for Burgers equation if we investigate the results of the rest of cases B – F. To make this point as clear as possible, let us discuss the results of case E. By using the results of case E and the special solutions of (5), we can obtain the following exact solutions from (11):

$$u_{17} = 2\nu kBa_{-1}^2\mathrm{e}^{B\eta}\left[\nu kb_0A^2(|a_{-1}|-a_{-1}) + a_{-1}|a_{-1}|\mathrm{e}^{B\eta}+\nu kb_0(|a_{-1}|+a_{-1})\mathrm{e}^{2B\eta}\right]^{-1}, \quad \lambda = \frac{\nu kBa_{-1}}{|a_{-1}|}, \quad (37)$$

$$u_{18} = 4\nu ka_{-1}\left[a_{-1}(\cos(2\eta)-1)+\mathrm{i}|a_{-1}|(1+\sin(2\eta))\right] \cdot \left[2\mathrm{i}\nu k|a_{-1}|b_0-\mathrm{i}a_{-1}|a_{-1}|-a_{-1}^2+[2\nu kb_0(\mathrm{i}|a_{-1}|-a_{-1}) - a_{-1}^2]\sin(2\eta)+[\mathrm{i}a_{-1}|a_{-1}|-2\nu kb_0(\mathrm{i}|a_{-1}| + a_{-1})]\cos(2\eta)\right]^{-1}, \quad \lambda = \frac{-\mathrm{i}\nu ka_{-1}}{|a_{-1}|}, \quad (38)$$

$$u_{19} = 4\nu ka_{-1}\left[a_{-1}(1-\cos(2\eta))+\mathrm{i}|a_{-1}|(\sin(2\eta)-1)\right] \cdot \left[2\mathrm{i}\nu k|a_{-1}|b_0-\mathrm{i}a_{-1}|a_{-1}|-a_{-1}^2+[2\nu kb_0(a_{-1}-\mathrm{i}|a_{-1}|) + a_{-1}^2]\sin(2\eta)+[\mathrm{i}a_{-1}|a_{-1}|-2\nu kb_0(\mathrm{i}|a_{-1}| + a_{-1})]\cos(2\eta)\right]^{-1}, \quad \lambda = \frac{\mathrm{i}\nu ka_{-1}}{|a_{-1}|}, \quad (39)$$

$$u_{20} = 4\nu ka_{-1}\left[-a_{-1}(1+\cos(2\eta))+\mathrm{i}|a_{-1}|(1-\sin(2\eta))\right] \cdot \left[2\mathrm{i}\nu k|a_{-1}|b_0+\mathrm{i}a_{-1}|a_{-1}|+a_{-1}^2+[2\nu kb_0(\mathrm{i}|a_{-1}|-a_{-1}) + a_{-1}^2]\sin(2\eta)+[\mathrm{i}a_{-1}|a_{-1}|+2\nu kb_0(\mathrm{i}|a_{-1}| + a_{-1})]\cos(2\eta)\right]^{-1}, \quad \lambda = \frac{\mathrm{i}\nu ka_{-1}}{|a_{-1}|}, \quad (40)$$

$$u_{21} = 4\nu ka_{-1}\left[a_{-1}(1+\cos(2\eta))+\mathrm{i}|a_{-1}|(\sin(2\eta)-1)\right] \cdot \left[2\mathrm{i}\nu k|a_{-1}|b_0+\mathrm{i}a_{-1}|a_{-1}|+a_{-1}^2+[2\nu kb_0(a_{-1}-\mathrm{i}|a_{-1}|)\right.$$

$$-a_{-1}^2]\sin(2\eta)+[\mathrm{i}a_{-1}|a_{-1}|+2\nu k b_0(\mathrm{i}|a_{-1}| +a_{-1})]\cos(2\eta)]^{-1}, \quad \lambda=\frac{-\mathrm{i}\nu k a_{-1}}{|a_{-1}|}. \qquad (41)$$

Here, it is also important to note that all the exact solutions given by (37 – 41) and obtained through the results of Case E can be verified by substitution. The main feature of these exact solutions is the inclusion of the free parameters a_{-1} and b_0. Moreover, to the best of the author's knowledge, all these exact solutions are new for Burgers equation. Furthermore, the generalized Exp-function method introduced in this paper not only gives more exact solutions compared with the original method but also has the advantage to be applied to nonlinear equations including those in which the odd and even-order derivative terms are coexist as seen from the example of Burgers equation.

5. Conclusion

The Exp-function method has been successfully generalized in this paper. As seen from the example of Burgers equation, the main advantage of this generalized method over the other methods is that it can be applied to a wide class of nonlinear evolution equations including those in which the odd and even-order derivative terms are coexist.

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